Quiz 3 Solution

Problem 1

$S_Z = \{1,9\}$

$P_Z(9) = P(X = -3) + P(X = 3) = ½$

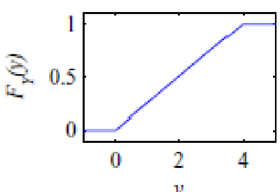
$P_Z(1) = P(X = -1) + P(X = 1) = ½$

$P_Z(z) = 0$, if $z \notin \{1,9\}$

$E[Z] = 1 * ½ + 9 * ½ = 10 /2 = 5.$

Problem 2

The CDF of Y is



$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/4 & 0 \le y \le 4 \\ 1 & y > 4 \end{cases} \quad (1)$$

From the CDF $F_Y(y)$, we can calculate the probabilities:

(1) $P[Y \le -1] = F_Y(-1) = 0$

(2) $P[Y \le 1] = F_Y(1) = 1/4$

(3) $P[2 < Y \le 3] = F_Y(3) - F_Y(2) = 3/4 - 2/4 = 1/4$

(4) $P[Y > 1.5] = 1 - P[Y \le 1.5] = 1 - F_Y(1.5) = 1 - (1.5)/4 = 5/8$

Problem 3

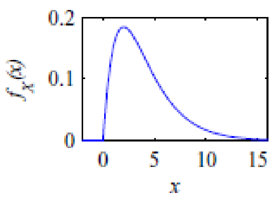
(1) First we will find the constant c and then we will sketch the PDF. To find c, we use the fact that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$. We will evaluate this integral using integration by parts:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} cxe^{-x/2}\,dx \tag{1}$$

$$= \underbrace{-2cxe^{-x/2}\Big|_0^{\infty}}_{=0} + \int_0^{\infty} 2ce^{-x/2}\,dx \tag{2}$$

$$= -4ce^{-x/2}\Big|_0^{\infty} = 4c \tag{3}$$

Thus $c = 1/4$ and X has the Erlang $(n = 2, \lambda = 1/2)$ PDF



$$f_X(x) = \begin{cases} (x/4)e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

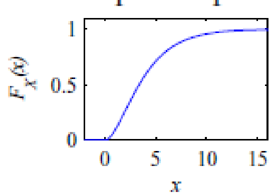
(2) To find the CDF $F_X(x)$, we first note X is a nonnegative random variable so that $F_X(x) = 0$ for all $x < 0$. For $x \geq 0$,

$$F_X(x) = \int_0^x f_X(y)\,dy = \int_0^x \frac{y}{4}e^{-y/2}\,dy \tag{5}$$

$$= -\frac{y}{2}e^{-y/2}\Big|_0^x - \int_0^x -\frac{1}{2}e^{-y/2}\,dy \tag{6}$$

$$= 1 - \frac{x}{2}e^{-x/2} - e^{-x/2} \tag{7}$$

The complete expression for the CDF is



$$F_X(x) = \begin{cases} 1 - \left(\frac{x}{2} + 1\right)e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

(3) From the CDF $F_X(x)$,

$$P[0 \leq X \leq 4] = F_X(4) - F_X(0) = 1 - 3e^{-2}. \tag{9}$$

(4) Similarly,

$$P[-2 \leq X \leq 2] = F_X(2) - F_X(-2) = 1 - 3e^{-1}. \tag{10}$$